Heat Equation

conducting rod length L

heat can flow along the rod

but not across lateral surface ("up" or "down")

temperature is $u(x,t)$ x: position

t: time

the governing equation is the Heat Equation

$$\boxed{\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}} \qquad \text{(1-D Heat eq.)}$$

or $\quad u_t = K u_{xx} \qquad (K>0)$

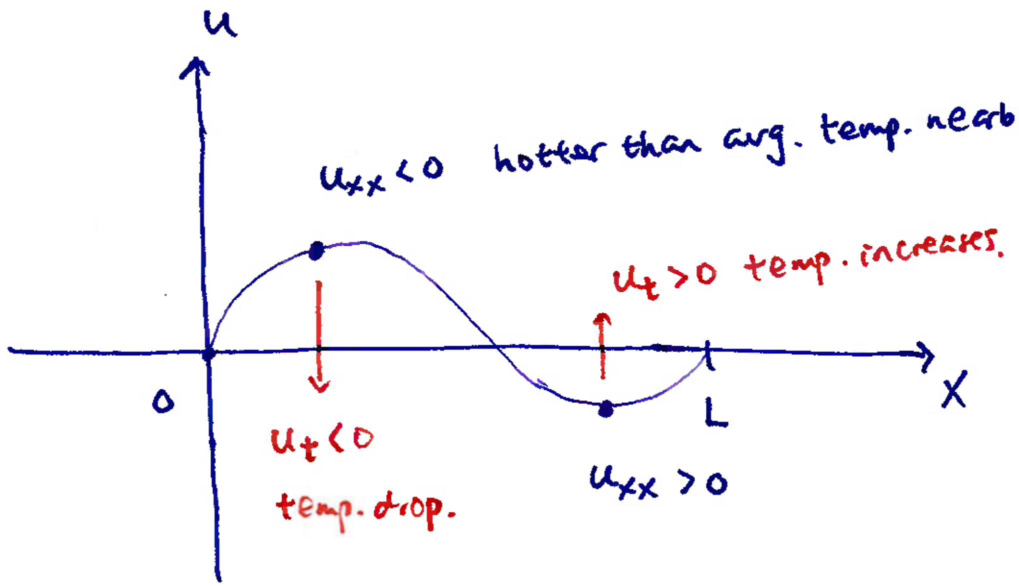
this is a partial differential equation (PDE)

the heat eq. comes from conservation of energy and Fourier's law what does $u_t = k u_{xx}$ say?

u_{xx} : concavity of $u(x,t)$ in space (position)

u_{xx} : concave up $\rightarrow$ $u_t > 0$ $\rightarrow$ temp. rises over time
($u_{xx} > 0$)

u_{xx} : concave down $\rightarrow$ $u_t < 0$ $\rightarrow$ temp. falls over time
($u_{xx} < 0$)



How to solve $u_t = k u_{xx}$?

Set up: $\quad 0 < x < L \qquad t > 0$

$$\left.\begin{aligned} u(0,t) &= T_1 \quad \text{(left end temp. for all } t\text{)} \\ u(L,t) &= T_2 \quad \text{(right end temp. for all } t\text{)} \end{aligned}\right\} \text{Boundary conditions (BC)}$$

$$u(x,0) = f(x) \quad \text{(initial temp. profile)} \qquad \text{Initial condition (IC)}$$

let's solve the case where $T_1 = T_2 = 0$ (homogeneous BC's)

method of separation of variables

$$u(x,t) = X(x)\,T(t)$$

product of two functions

X : function of x only

T : " " t "

$$u_t = \frac{\partial}{\partial t}\Big(\underbrace{X(x)}_{\text{"constant"}}T(t)\Big) = X T'$$

likewise, $u_{xx} = X'' T$

sub into $u_t = k u_{xx}$

$$X T' = k X'' T \qquad \text{separate the functions}$$

$$\underbrace{\frac{X''}{X}}_{\substack{\text{function of} \\ x \text{ only}}} = \underbrace{\frac{T'}{kT}}_{\substack{\text{function of} \\ t \text{ only}}} = \text{constant} = -\lambda \qquad (\lambda > 0)$$

separation constant

$$\frac{X''}{X} = -\lambda \implies \boxed{X'' + \lambda X = 0}$$

$$\frac{T'}{kT} = -\lambda \implies \boxed{T' + \lambda k T = 0}$$

$\Big\}$ Two ODE's

BC's: $u(0,t) = 0 \to X(0)T(t) = 0 \to \boxed{X(0) = 0}$

$u(L,t) = 0 \to X(L)T(t) = 0 \to \boxed{X(L) = 0}$

let's solve $X'' + \lambda X = 0$ $X(0) = X(L) = 0$

$$X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

$$X(0) = A = 0$$

$$X(L) = B\sin(\sqrt{\lambda}\,L) = 0$$ require $B \neq 0$

$$\sin(\sqrt{\lambda}\,L) = 0$$

$$\sqrt{\lambda}\,L = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$$

for each n there is one solution

eigenvalues

so ~~one~~ each n, there is one solution

$$\boxed{X_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$$

(drop B scaling constant)

eigenfunctions

now $T' + k\lambda T = 0$ use $\lambda = \frac{n^2\pi^2}{L^2}$

$$T' + \frac{kn^2\pi^2}{L^2} T = 0$$

$$T(t) = C e^{-kn^2\pi^2/L^2 \, t}$$ one for each $n = 1, 2, 3, \ldots$

$$\boxed{T_n(t) = e^{-kn^2\pi^2/L^2 \, t}}$$

since $u(x,t) = X(x)\,T(t)$

for each n, there is one solution

$$u_n(x,t) = X_n T_n$$

general solution is linear combo of all

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{-kn^2\pi^2/L^2 \, t} \sin\left(\frac{n\pi x}{L}\right)}$$

last unused condition: $u(x,0) = f(x)$ (initial condition)

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$ sine series half period L

$$\boxed{C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$